AD-A097 989   TEL-AVIV UNIV (ISRAEL) DEPT OF APPLIED MATHEMATICS   F/G 12/1
THE STABILITY OF PSEUDOSPECTRAL-CHEBYSHEV METHODS.(U)
1981   D GOTTLIEB   AFOSR-77-3405
UNCLASSIFIED   AFOSR-TR-81-0390   NL

1 OF 1

END
DATE
FILMED
5-81
DTIC

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LEVEL II

(3)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR-81-0311

AD-A097989

4. TITLE (and Subtitle): THE STABILITY OF PSEUDOSPECTRAL-CHEBYSHEV METHODS

5. TYPE OF REPORT & PERIOD COVERED: INTERIM rept.

7. AUTHOR(s): David Gottlieb

8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3405 New

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Department of Applied Mathematics
Tel-Aviv University
Tel-Aviv, Israel

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2304/A3 61102F

11. CONTROLLING OFFICE NAME AND ADDRESS:
Air Force Office of Scientific Research/NM
Bolling AFB DC 20332

12. REPORT DATE: 1981

13. NUMBER OF PAGES: 27

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

(17) A3 (12) 24

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

DTIC
APR 2 2 1981

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The stability of pseudospectral-Chebyshev methods is demonstrated for parabolic and hyperbolic problems with variable coefficients. The choice of collocation points is discussed. Numerical examples are given for the case of variable coefficient hyperbolic equations.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AFOSR-TR- 81-0390

THE STABILITY OF PSEUDOSPECTRAL-CHEBYSHEV METHODS

David Gottlieb

Department of Applied Mathematics
Tel-Aviv University, Tel-Aviv, Israel

ABSTRACT

The stability of pseudospectral-Chebyshev methods is demonstrated for parabolic and hyperbolic problems with variable coefficients. The choice of collocation points is discussed. Numerical examples are given for the case of variable-coefficient hyperbolic equations.

X

A

This report was supported by the Air Force Office of Scientific Research under Grant No. [redacted]. AFOSR-77-3405

81 4 22 106

1. Introduction

The purpose of this paper is to analyze spectral Chebyshev collocation (otherwise known as pseudospectral) methods for hyperbolic and parabolic problems. We shall show that these methods converge at a rate that is faster than that of finite differences. The analysis is based upon results presented in [1]. This reference outlines the general theory of convergence of spectral methods and proves that if a spectral method is algebraically stable in some norm, then the method is strongly stable in an algebraically equivalent new norm. If in addition the method is consistent by virtue of its truncation error tending to zero in this new norm, then convergence is implied.

The application of this theory to hyperbolic and parabolic problems had been discussed in [1] mainly for constant coefficient hyperbolic and parabolic problems and, in the case of Chebyshev methods, mainly for the Galerkin and Tau methods. In this paper we discuss the collocation methods and prove stability for the variable coefficient case. The new idea that enables us to establish stability for collocation methods is a new quadrature formula for Gauss-type integration. We use the positive weights given by this formula as the new norm and prove energy conservation in this norm. Using the same technique a new proof is presented for variable coefficient hyperbolic and parabolic problems when solved by spectral-Chebyshev methods using Tau methods. These proofs are more general than those in [1] in the sense that they include the variable coefficient case.

Section 1

A numerical solution of the problem

$$u_t = Lu \tag{1.1}$$

where $u \in H$, H is an Hilbert space and L is an infinite dimensional operator, consists of two steps. The first is to choose a finite dimensional subspace of H, say B_N, and the second is to choose a projection operation $P_N : H \to B_N$. The approximation to (1.1) becomes

$$\frac{\partial u_N}{\partial t} = P_N L P_N u_N \qquad u_N \in B_N \tag{1.2}$$

which may be solved on a computer. Spectral Chebyshev methods are defined by choosing B_N as the N-dimensional space spanned by polynomials of degree $N+k-1$ that satisfy boundary conditions.

There are three ways which have been used to choose the operator P_N, namely Galerkin, Tau and collocation.

In the Galerkin method for homogeneous boundary conditions we choose ϕ_n $n = 1,\ldots,N$ as the basis of B_N and solve

$$\left(\frac{\partial u_N}{\partial t} - L u_N, \phi_n \right) = 0 \qquad n = 1,\ldots,N$$

$$u_N = \sum_{n=1}^{N} a_n \phi_n . \tag{1.3}$$

For the Tau method we choose $\{\phi_n\}$ to be a set of orthogonal functions such that $(\phi_n,\phi_m) = \delta_{nm}$ and expand

$$u_N = \sum_{n=1}^{N+k} a_n \phi_n$$

where k is the number of boundary conditions. Then set

$$\left(\frac{\partial u_N}{\partial t} - L\, u_N,\ \phi_n \right) = 0 \qquad n = 1,\ldots,N \ . \tag{1.4}$$

The condition $u_N \in B_N$ provides the other k equations.

In the collocation method we set

$$u_N = \sum_{n=1}^{N} a_n \phi_n$$

and require

$$\frac{\partial u_N}{\partial t} - L\, u_N = 0 \qquad \text{for } x_j \qquad j = 1,\ldots,N \ . \tag{1.5}$$

It had been observed by Orszag [1] and Kreiss and Oliger [2] that the collocation method can be carried out efficiently in the physical space in contrast to the Galerkin and Tau methods which must be solved in the transform space. This fact enables one to use the collocation method efficiently for nonlinear equations. We refer the reader to [1] for further discussion of this fact.

In the next sections we will illustrate the above procedure applied to parabolic and hyperbolic equations.

Section 2

Consider the equation

$$\begin{aligned} & u_t = S(x)\, u_{xx} \qquad && -1 \le x \le 1 \\ & && 0 < \delta < S(x) \\ & u(\pm 1, t) = 0 \ . && \end{aligned} \tag{2.1}$$

In the Galerkin-Chebyshev method we choose

$$\begin{aligned} \phi_n &= T_n - T_0 \qquad n \text{ even}, \\ \phi_n &= T_n - T_1 \qquad n \text{ odd}. \end{aligned} \tag{2.2}$$

where $T_n(x) = \cos(n \cos^{-1} x)$.

We expand $u_N = \sum_{n=0}^{N} a_n \phi_n(x)$ so that $u_N(\pm 1, t) = 0$ and set

$$\int_{-1}^{1} \left[\frac{\partial u_N}{\partial t} - S(x) \frac{\partial^2 u_N}{\partial x^2} \right] \frac{\phi_n}{\sqrt{1 - x^2}} \, dx = 0 \qquad n = 2, \ldots, N \,. \tag{2.3}$$

It is readily seen that for nonconstant $S(x)$, it is difficult to solve the equations for (2.3). Orszag has found some efficient transform methods to evaluate

$$\int_{-1}^{1} S(x) \frac{\partial^2 u_N}{\partial x^2} \frac{\phi_n}{\sqrt{1 - x^2}} \,.$$

In general, however, solving (2.3) for the coefficients $\{a_n\}$ is time consuming.

In the Tau method we set

$$u_N = \sum_{n=0}^{N+2} a_n T_n(x)$$

and require

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \left(\frac{\partial u_N}{\partial t} - S(x) \frac{\partial^2 u_N}{\partial x^2} \right) T_n \, dx = 0 \qquad n = 0, \ldots, N$$

together with

$$\sum_{n=0}^{N+2} a_n T_n(1) = \sum_{n=0}^{N+2} a_n = 0$$

and

$$\sum_{n=0}^{N+2} a_n T_n(-1) = \sum_{n=0}^{N+2} (-1)^n a_n = 0 \ . \tag{2.4}$$

We face the same complications for getting the coefficients as we had for the Galerkin method.

In the collocation method we set

$$u_N = \sum_{n=0}^{N} a_n \phi_n(x)$$

where the ϕ_n's are defined in (2.2). Then we demand

$$\frac{\partial u_N}{\partial t} - S(x) \frac{\partial^2 u_N}{\partial x^2} = 0 \quad \text{at} \quad x = x_j \qquad j = 0,\dots N \tag{2.5}$$

for some points x_j. If the x_j are chosen to be $\cos \frac{\pi j}{N}$ so that the boundary values are included, there is an efficient way to solve (2.5), by taking advantage of the orthogonality of the trigonometric functions. Set

$$u_N(x) = \sum_{n=0}^{N} a_n T_n(x) \ , \quad u_N(x_j) = \sum_{n=0}^{N} a_n T_n(x_j) \ , \quad 0 \le j \le N$$

Then

$$a_n = \frac{2}{Nc_n} \sum_{k=0}^{N} \frac{1}{c_k} u_N(x_k) \cos \frac{\pi nk}{N} \qquad \begin{array}{l} c_0 = c_N = 2 \\ c_k = 1 \quad 1 \le k \le N-1 \end{array} \qquad (2.5a)$$

Using the properties of Chebyshev polynomials we may set

$$\frac{\partial^2 u_N(x_j)}{\partial x^2} = \sum_{n=0}^{N} b_n T_n(x_j)$$

where the coefficients b_n may be found from

$$c_n b_n = \sum_{\substack{p = n+2 \\ p+n \text{ even}}}^{N} p(p^2-n^2)a_p \ .$$

Then we go back to the physical space and solve

$$\frac{\partial u_N}{\partial t}(x_j) = S(x_j) \frac{\partial^2 u_N}{\partial x^2}(x_j) \qquad j = 1,\ldots,N-1$$

$$u_N(x_0) = u_N(x_N) = 0 \ .$$

This procedure is very efficient and may be generalized without any problem to nonlinear equations. In practice we would use the Chebyshev polynomials to interpolate u spatially and then to evaluate the spatial derivative at the desired points x_j. Finally the solution would be advanced in time using the original nonlinear equation to find the time derivative at the points x_j in the physical space.

In order to prove convergence we need the following two results.

Lemma 1.

Let u satisfy $u(\pm 1) = 0$ and have a continuous first derivative, then

$$\int_{-1}^{1} \frac{uu_{xx}}{\sqrt{1-x^2}}\, dx \le 0 \quad . \tag{2.6}$$

For the proof we refer the reader to [1, p. 82].

Lemma 2.

Let $x_j = \cos \frac{\pi j}{N}$ $j = 0,\ldots,N$. Then there exist $w_j > 0$, $j=0,\ldots N$ such that

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx = \sum_{j=0}^{N} f(x_j)w_j \tag{2.7}$$

$$\sum_{j=0}^{N} w_j = \pi$$

for any $f(x)$ which is a polynomial of degree at most $2N-1$.

Remark: The formula (2.7) is a generalization of the well known integration rule of Gauss type. Note that w_j depends on N. There are two major differences between (2.7) and the usual Gauss integration formula for the weight function $w(x) = (1-x^2)^{-\frac{1}{2}}$. The first difference is that the Gauss formulas are of open type, that is the boundary points are not included, whereas in (2.7) they are included. The second is that the interior points are not the zeroes of the orthogonal polynomials with respect to $w(x) = (1-x^2)^{-\frac{1}{2}}$ namely the Chebyshev polynomial of the first kind, but rather that they are the zeroes of the Chebyshev polynomial of the second kind (which are orthogonal with respect to the weight function $w(x) = (1-x^2)^{\frac{1}{2}}$). It is interesting to note that Lemma 2 implies the Gauss integration formula for the weight function $w(x) = (1-x^2)^{\frac{1}{2}}$, i.e., that the formula

$$\int_{-1}^{1} g(x)\sqrt{1-x^2} = \sum_{j=1}^{N-1} g(x_j)\ \hat{w}_j \qquad (2.8)$$

is correct for any $2N-3$ degree polynomial. In fact if (2.7) is correct set $f(x) = (1-x^2)g(x)$ in (2.7) to get

$$\int_{-1}^{1} g(x)\sqrt{1-x^2} = \sum_{j=0}^{N} (1-x_j^2)g(x_j)\ w_j = \sum_{j=1}^{N-1} (1-x_j^2)g(x_j)\ w_j$$

since $x_0 = 1$, $x_N = -1$, and the fact that if $g(x)$ is a $2N-3^{rd}$ degree polynomial then the degree of f is $2N-1$. Equation (2.8) is now established with $\hat{w}_j = (1-x_j^2)\ w_j$.

Proof:

We first note that (2.7) can be made exact for any N^{th} degree polynomial by putting $f(x) = x^n$, $n = 0,\ldots,N$ and solving for the $N+1$ unknowns, w_j since the Vandermonde matrix is nonsingular.

Let $f(x)$ be now a polynomial of degree $2N-1$. Then there are $g(x)$ of degree $N-2$ and $v(x)$ of degree N such that

$$f(x) = (1-x^2)Y_{N-1}(x)g(x) + v(x) \qquad (2.9)$$

where Y_{N-1} is the Chebyshev polynomial of the second kind, i.e.,

$$Y_{N-1} = \frac{\sin(N\cos^{-1}x)}{\sqrt{1-x^2}} \quad . \qquad (2.10)$$

Since

$$Y_{N-1}(x_j) = \frac{\sin(\pi j)}{\sin\frac{\pi j}{N}} = 0 \qquad j=1,\ldots N-1$$

we conclude that

$$f(x_j) = v(x_j) \qquad j=0,\ldots,N \quad . \tag{2.11}$$

Now

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \sqrt{1-x^2}\; Y_{N-1}(x) g(x)\,dx + \int_{-1}^{1} \frac{v(x)}{\sqrt{1-x^2}}\,dx \quad . \tag{2.12}$$

The first term in the right hand side of (2.12) vanishes since Y_{N-1} is orthogonal to any polynomial of degree less than N-1 and the degree of $g(x)$ is N-2. Since $v(x)$ is a polynomial of degree N (2.7) is exact for it and therefore

$$\int_{-1}^{1} \frac{v(x)}{\sqrt{1-x^2}}\,dx = \sum_{j=0}^{N} v(x_j)\, w_j$$

and by (2.11) we conclude that

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{v(x)}{\sqrt{1-x^2}}\,dx = \sum_{j=0}^{N} v(x_j)\, w_j = \sum_{j=0}^{N} f(x_j)\, w_j \quad .$$

This proves (2.7) except for the fact that $w_j > 0$. To prove that we define

$$f_\ell(x) = (1-x^2) Y_{N-1}^2(x) / [Y'_{N-1}(x_\ell)(x-x_\ell)]^2 \qquad \ell = 1,\ldots,N-1 \quad .$$

The degree of $f_\ell(x)$ is 2N-2; moreover, $f_\ell(x_j) = \delta_{\ell j}(1-x^2)$ and $f_\ell(x) \geq 0$, $-1 \leq x \leq 1$. Therefore (2.7) is exact and yields

$$\int_{-1}^{1} \frac{f_\ell(x)}{\sqrt{1-x^2}}\, dx = \sum_{j=0}^{N} f_\ell(x_j)\, w_j = (1-x^2)\, w_\ell . \tag{2.13}$$

Equation (2.13) shows that $w_\ell > 0$, $\ell = 1,\ldots,N-1$. Define now

$$f_0(x) = (1+x)\, \frac{Y_{N-1}^2(x)}{2N^2} \geq 0 \qquad \text{(a)}$$

and

$$f_N(x) = (1-x)\, \frac{Y_{N-1}^2}{2N^2} \geq 0 \qquad \text{(b)} \tag{2.14}$$

$$f_0(x_j) = \delta_{0j} \qquad f_N(x_j) = \delta_{Nj}$$

and therefore

$$w_0 = \int_{-1}^{1} \frac{f_0(x)}{\sqrt{1-x^2}}\, dx > 0 \qquad \text{(a)}$$

$$w_N = \int_{-1}^{1} \frac{f_N(x)}{\sqrt{1-x^2}}\, dx > 0 \qquad \text{(b)} \tag{2.15}$$

and this concludes the proof.

We are now ready to prove the stability of the Chebyshev collocation method for the heat equation.

<u>Theorem</u>: (Stability)

Let u_N be the Chebyshev collocation approximation (2.5) to the heat equation (2.1). Then

$$\sum_{j=0}^{N} \frac{u_N^2(x_j,t)}{S(x_j)}\, w_j \leq \sum_{j=0}^{N} u_N^2(x_j,0)\, \frac{w_j}{S(x_j)} . \tag{2.16}$$

Proof:

Since

$$\frac{\partial u_N}{\partial t}(x_j,t) = S(x_j)\,\frac{\partial^2 u_N}{\partial x^2}(x_j) \qquad j=1,\ldots,N-1 \quad \text{and}$$

and

$$u_N(x_0) = u_N(x_N) = 0.$$

We get

$$\sum_{j=0}^{N} u_N(x_j)\,\frac{\partial u_N}{\partial t}(x_j)\,\frac{w_j}{S(x_j)} = \sum_{j=0}^{N} u_N(x_j)\,\frac{\partial^2 u_N}{\partial x^2}(x_j)\,w_j \;. \tag{2.17}$$

By Lemma 2.2 and the fact that the degree of $u_N \dfrac{\partial^2 u_N}{\partial x^2}$ is 2N-2 we get

$$\sum_{j=0}^{N} u_N(x_j)\,\frac{\partial^2 u_N}{\partial x^2}(x_j)\,w_j = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,u_N\,\frac{\partial^2 u_N}{\partial x^2}\,dx \le 0 \;.$$

The last inequality follows from Lemma 2, since $u_N(\pm 1,t) = 0$. Therefore,

$$\frac{d}{dt}\sum_{j=0}^{N} u_N^2(x_j)\,\frac{w_j}{S(x_j)} \le 0$$

and (2.16) follows.

The next step for showing convergence is to show that the truncation error tends to zero as N^{-1} tends to infinity. In view of the discussion in [1, p. 48] the truncation error is given by

$$\left\| \left(P_N\,S(x)\,\frac{\partial^2}{\partial x^2}\,P_N - P_N S(x)\,\frac{\partial^2}{\partial x^2}\right) u \right\| \tag{2.18}$$

where $u \in C^\infty$ is the solution to (2.1), $P_N f(x)$ is a polynomial of degree N that interpolates the function $f(x)$ at the points x_j and

$$\| g \| = \left(\sum_{j=0}^{N} \frac{g^2(x_j)}{S(x_j)} w_j \right)^{\frac{1}{2}} \qquad x_j = \cos \frac{\pi j}{N} \qquad j=0,\ldots,N$$

Theorem: (Consistency)

Let u, P_N and $\| \cdot \|$ be defined as above then

$$\| P_N \, S(x) \frac{\partial^2}{\partial x^2} P_N - S(x) \frac{\partial^2}{\partial x^2} u \, \| = 0\left(\frac{1}{N^r}\right) \tag{2.19}$$

for any positive r.

Proof:

From (2.5a) we can express $P_N u$ by

$$P_N u = \sum_{n=0}^{N} \frac{\alpha_n}{c_n} T_n(x)$$

where

$$\alpha_n = \frac{2}{N} \sum_{j=0}^{N} \frac{u(x_j)}{c_j} T_n(x_j) \quad .$$

On the other hand

$$u(x) = \frac{a_0}{2} T_0 + \sum_{n=1}^{\infty} a_n T_n(x)$$

where

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{u(x) T_n(x)}{\sqrt{1-x^2}} \, dx \quad .$$

It is well known that $a_n = 0\left(\frac{1}{n^p}\right)$ for any p. Moreoever α_n can be expressed in terms of the a_n's by the formula

$$\alpha_n = \frac{1}{N} a_0 \sum_{j=0}^{N} \frac{T_k(x_j) T_n(x_j)}{i_j} + \frac{2}{N} \sum_{k=1}^{N} a_k \sum_{j=0}^{N} \frac{T_k(x_j) T_n(x_j)}{i_j}$$

$$= a_n + a_{2N-n} + a_{4N-n} + \ldots.. \tag{2.20}$$

and therefore

$$S(x) \frac{\partial^2}{\partial x^2} P_N u - S(x) \frac{\partial^2}{\partial x^2} u = S(x) \sum_{n=0}^{N} (a_n - a_n) T_n'' - S(x) \sum_{n=N}^{\infty} a_n T_n'' \tag{2.21}$$

define

$$g(x) = S(x) \sum_{n=N}^{\infty} a_n T_n''$$

then

$$\| g(x) \|^2 = \sum_{j=0}^{N} S(x_j) \left(\sum_{n=N}^{\infty} a_n T_n'' (x_j) \right)^2 w_j = 0\left(\frac{1}{N^r}\right) .$$

Moreover since $|\alpha_n - a_n| = 0\left(\frac{1}{N^r}\right)$ by (2.20) we get

$$\left\| S(x) \frac{\partial^2}{\partial x^2} P_N u - S(x) \frac{\partial^2}{\partial x^2} u \right\| = O\left(\frac{1}{N^r}\right)$$

and since $\|P_N\| = 1$ (2.19) is proven.

Section 3

In this section we would like to treat the hyperbolic equation

$$u_t = S(x)u_x \qquad S(x) > 0 \qquad |x| \le 1$$

$$u(1,t) = 0 . \tag{3.1}$$

We concentrate upon the collocation method. There are currently two ways of performing the collocation method. The first one is to collocate at the point $x_k = \cos \frac{\pi k}{N}$, $k=1,\ldots,N$ and to use the boundary condition for $x_0 = 1$. This means that we collocate at $N-1$ points in the interior of the domain and also at the outflow boundary; we do not collocate at $x = 1$ since a boundary condition is imposed at this point. The other way is to collocate at the points $x_k = \cos \frac{\pi k}{N}$, $k=1,\ldots,N-1$ and to use the boundary condition at $x_0 = 1$. This amounts to using $N-1$ interior points for collocation and to impose a boundary condition at the inflow. The outflow boundary is not treated at all. We would now like to show how to carry out these two methods effectively.

In order to carry out the first one we expand

$$u_N(x_n,t) = \sum_{k=0}^{N} a_k T_k(x_n) \qquad \begin{array}{l} n = 0,\ldots,N \\ x_n = \cos \frac{\pi n}{N} \end{array} \tag{3.2}$$

and solve for a_k

$$a_k = \frac{1}{c_k} \sum_{j=0}^{N} \frac{1}{c_j} u_N(x_j,t) \cos \frac{\pi jk}{N} \qquad \begin{matrix} c_0 = c_n = 2 \\ c_\ell = 1 \quad 0 \neq \ell \neq N \end{matrix} \quad . \tag{3.3}$$

Equation (3.3) is evaluated by using the Fast Fourier Transform (FFT) method.

Now

$$\frac{\partial u_N}{\partial x}(x_n,t) = \sum_{k=0}^{N} b_k T_k(x_n) \ , \tag{3.4}$$

where

$$b_k = \frac{1}{c_k} \sum_{\substack{k=p+1 \\ k+p \text{ odd}}}^{N} 2pa_p \quad . \tag{3.5}$$

The evaluation of the right hand side of equation (3.4) is carried out using FFT. Then equation (3.5) is solved for the b_k's with $0(N)$ operations, that is a simple recursive formula is used

$$b_N = 0 \qquad b_{N-1} = 2Na_n$$

and

$$b_{k+2} - b_k = \frac{1}{c_{k+2}} 2(k+1)a_{k+1} \quad .$$

Then we solve in the physical space.

$$\begin{aligned} &\frac{\partial u_N}{\partial t}(x_j,t) = S(x_j)\frac{\partial u_N}{\partial x}(x_j,t) \qquad j = 1,\ldots,N \\ &u_N(1,t) = 0 \quad . \end{aligned} \tag{3.6}$$

A very efficient time marching technique which is explicit and unconditionally stable had been developed in [3] and can be used for the solution of (3.6).

The second way of collocation is carried out as follows. Set

$$v_N(x_n,t) = \sum_{k=0}^{N-1} d_k \, T_k(x_n) \qquad \begin{array}{l} n = 0,\ldots,N-1 \\ x_n = \cos \frac{\pi n}{N} \end{array} \quad . \tag{3.7}$$

It can be shown that d_k can be expressed in terms of a_k derived in (3.3). In fact

$$e_k = a_k + (-1)^{N-1} 2a_N \frac{(-1)^k}{c_k} \quad . \tag{3.8}$$

Equation (3.8) is derived as follows

$$\begin{aligned} u_N(x_n,t) &= \sum_{k=0}^{N-1} a_k \, T_k(x_n) + a_N T_N(x_n) \\ &= \sum_{k=0}^{N-1} a_k \, T_k(x_n) + (-1)^{N-1} 2a_N \left[\sum_{k=0}^{N} \frac{(-1)^k}{c_k} T_k(x_n) - \frac{(-1)^N}{c_N} T_N(x_n) \right] \\ &= \sum_{k=0}^{N-1} a_k \, T_k(x_n) + (-1)^{N-1} 2a_N \sum_{k=0}^{N-1} \frac{(-1)^k}{c_k} T_k(x_n) = \sum_{k=0}^{N-1} e_k \, T_k(x_n) \end{aligned}$$

for $n = 0,\ldots,N-1$.

Now

$$\frac{\partial v_N}{\partial x}(x_n,t) = \sum_{k=0}^{N-1} \gamma_k \, T_k(x_n)$$

where

$$\gamma_k = \frac{1}{c_k} \sum_{\substack{k=p+1 \\ k+p \text{ odd}}}^{N} 2p \, e_p \tag{3.9}$$

and we solve

$$\frac{\partial v_N(x_j,t)}{\partial t} = S(x_n)\,\frac{\partial v_N(x_j,t)}{\partial x} \qquad j=1,\ldots,N-1 \qquad (3.10)$$

$$u_N(x_0,t) = 0 \ .$$

Observe that u_N in the second way of collocation (3.7) - (3.10) is a polynomial of degree N-1, whereas in (3.2) - (3.6) it is a polynomial of degree N. The similarity between these two different methods can be seen in the case where $S(x) = 1$. Since $\frac{\partial u_N}{\partial t}\ \frac{\partial u_N}{\partial x}$ is a polynomial of degree N that vanishes at $x = -1$ and at the series of $T_N'(x)$ we get

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} + \tau_1 (1+x) T_N'(x) \qquad (a)$$

and by the same argument (3.11)

$$\frac{\partial v_N}{\partial t} = \frac{\partial v_N}{\partial x} + \tau_2\, T_N'(x). \qquad (b)$$

It is interesting to note that for the Tau method one gets the error equation

$$\frac{\partial R_N}{\partial t} = \frac{\partial R_N}{\partial x} + \tau_3\, T_N(x). \qquad (3.12)$$

where R_N is the Tau approximation to u. It seems that the Tau method can be viewed, in the case of the constant coefficient problem (3.12) as a collocation method based on the collocation points

$$z_k = \cos\left(\frac{\pi}{2}\,\frac{2k-1}{N}\right) \qquad k=1,\ldots,N\ . \qquad (3.12a)$$

This observation suggests a convenient way of using the Tau method for the variable coefficient case as well, namely set

$$\frac{\partial R_N}{\partial t} - S(x)\frac{\partial R_N}{\partial x} = 0 \qquad x = z_k \qquad k = 1,\ldots,N \quad . \tag{3.13}$$

This method reduces in the constant coefficient case to the Tau method. In order to establish stability for the collocation method described in (3.7) - (3.10) we need the following Lemma:

<u>Lemma</u>

Let $x_k = \cos\frac{\pi k}{N}$, $k = 0,\ldots,N-1$ then the quadrature formula

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\; f(x)dx = \sum_{k=0}^{N-1} f(x_k)\,\hat{w}_k \tag{3.14}$$

where

$$\hat{w}_k = \frac{1}{(1-x_k)[Y_N'(x_k)]^2}\int_{-1}^{1}\frac{Y_{N-1}^2(x)}{(x-x_k)^2}(1-x)\sqrt{\frac{1+x}{1-x}}\;dx > 0$$

$$\hat{w}_0 = \frac{1}{Y_{N-1}^2(1)}\int_{-1}^{1} Y_{N-1}^2(x)\sqrt{\frac{1+x}{1-x}}\;dx > 0$$

is correct for every polynomial of degree $2N-2$ or less.

<u>Proof</u>.

Let $f(x)$ be a polynomial of degree $2N-2$. Set $g(x) = (1+x)f(x)$. Since $g(x)$ is a polynomial of degree $2N-1$, formula (2.7) is exact.

$$\int_{-1}^{1}\frac{(1+x)f(x)}{\sqrt{1-x^2}}\,dx = \sum_{j=0}^{N}(1+x_j)w_j f(x_j) = \sum_{j=0}^{N-1}(1+x_j)w_j\; f(x). \tag{3.15}$$

Equation (3.15) implies (3.14) and w_k can be derived by a standard argument.

Now let v_N be the collocation approximation to u gotten by (3.7) - (3.10). Then

$$\frac{\partial u_N}{\partial t} = c(x)\,\frac{\partial v_N}{\partial x} \qquad x = x_n \qquad n = 1,\ldots,N-1 \,. \tag{3.16}$$

Multiplying by $\frac{v_N(x_n)\hat{w}_n}{c(x_n)}$ we get from (3.16)

$$\sum_{n=0}^{N-1} v_N(x_n)\,\frac{\partial v_N}{\partial t}(x_n)\,\frac{\hat{w}_n}{c(x_n)} = \sum_{n=0}^{N-1} v_N(x_n)\,\frac{\partial v_N}{\partial x}(x_n)\,\hat{w}_n$$

and by (3.14)

$$= \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; v_N\,\frac{\partial u_N}{\partial x}\,dx \le \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\; v_N^2\,\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} \frac{v_N^2}{(1-x)\sqrt{1-x^2}}\,dx \,. \tag{3.17}$$

The boundary term in the right hand side of (3.17) vanishes since $v_N(1) = 0$ and v_N is a polynomial and therefore

$$\frac{d}{dt}\sum_{n=0}^{N-1} v_N^2(x_n)\,\frac{\hat{w}_n}{c(x_n)} \le 0 \tag{3.18}$$

or

$$\sum_{n=0}^{N-1} v_N^2(x_n,t)\,\frac{\hat{w}_n}{c(x_n)} \le \sum_{n=0}^{N-1} v_N^2(x_n,0)\,\frac{\hat{w}_n}{c(x_n)} \,.$$

From the definition of w_n it follows that the norm described by the weights $\frac{\hat{w}_n}{c(x_n)}$ is algebraically equivalent to the norm in which we have consistency, therefore algebraic stability is proved. The same idea can be utilized in showing the stability of the Tau method. In fact from (3.12) it is evident that

$$\sum_{j=1}^{N}(1+z_j)\frac{y_j}{1-z_j}R_N(z_j)\frac{\partial R_N}{\partial t}(z_j) = \sum_{j=1}^{N}\frac{y_j}{1-z_j}R_N(z_j)\frac{\partial R_N(z_j)}{\partial x}(1+z_j) \qquad (3.19)$$

where y_j are the weights in the Gauss-Chebyshev integration. From (3.17) it follows that

$$\frac{d}{dt}\sum_{j=1}^{N}\frac{y_j}{1-z_j}R_N^2(z_j,t) = \int_{-1}^{1}\frac{R_N}{1-x}\frac{\partial R_N}{\partial x}\frac{1+x}{\sqrt{1-x^2}}dx = -\int_{-1}^{1}\frac{R_N^2}{(1-x)^2}\frac{1}{\sqrt{1-x^2}}dx \le 0 \qquad (3.20)$$

which proves algebraic stability.

The stability of the collocation method described by (3.3) - (3.6) follows immediately from that described in (3.7) - (3.10). It can be seen from the relation (3.8). In fact, setting

$$\frac{\partial u_N}{\partial t} - \frac{\partial u_N}{\partial x} = (1+x)\left(\frac{\partial w_N}{\partial t} - \frac{\partial w_N}{\partial x}\right)$$

one gets (3.11)(a) from (3.11)(b). This completes the discussion of collocation method for scalar equations. We refer the reader to [4] in which proper ways of implementing spectral methods for systems is discussed.

Section 4

The proofs presented in the last section were confined to the case in which $c(x)$ does not change sign. This might be a weakness of the theory rather

than that of the collocation method. Numerical experiments using the pseudospectral methods have indicated that there is no instability, that is they show the solution does not grow with N even when $S(x)$ changes sign. There might be problems owing to growth in time of the solution or to the existence of a stationary characteristic in the neighborhood of either the boundary or some interior point. But these problems seem to occur because of lack of spatial resolution and not because of stability. In order to illustrate this fact let us consider two equations:

$$\begin{aligned} u_t &= -x\,u_x \qquad |x| \leq 1 \\ u(x,0) &= f(x) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} u_t &= x\,u_x \qquad |x| \leq 1 \\ u(x,0) &= f(x) \\ u(\pm 1,t) &= 0 \end{aligned} \tag{4.2}$$

We attempt to solve (4.1) and (4.2) by the Chebyshev collocation method. According to the popular belief, there should be instabilities in the solution since x changes sign in the domain. However, as indicated in Table I below no such instabilities were found. As a matter of fact we can prove

Theorem:

The Chebyshev collocation method for (4.1) is stable.

Proof:

Let u_N be the Chebyshev approximation to u gotten by the collocation method. Then

$$\frac{\partial u_N}{\partial t} + x\frac{\partial u_N}{\partial x} = 0 \qquad x = \cos\frac{\pi j}{N} \quad j=0,\ldots,N \qquad \text{(a)} \tag{4.3}$$

as in (3.2). Or

$$\frac{\partial v_N}{\partial t} + x\frac{\partial v_N}{\partial x} = 0 \qquad x = \cos\frac{\pi j}{N} \quad j=1,\ldots,N-1 \quad \text{(b)} \quad (4.3)$$

as in (3.7). Since u_N is N^{th} degree polynomial and v_N is $N-1$ degree polynomial, we get

$$\frac{\partial w_N}{\partial t} = -x\frac{\partial w_N}{\partial x} \qquad -1 \le x \le 1 \qquad (4.4)$$

where w is either u_N or v_N.

We now refer the reader to ([1], p. 85-87) for the proof that (4.4) implies stability.

Theorem:

The Chebyshev collocation approximation for (4.2) is stable.

Proof:

Now u_N satisfies

$$\frac{\partial u_N}{\partial t} = x\frac{\partial u_N}{\partial x} \qquad x = \cos\frac{\pi j}{N} \qquad j=1,\ldots,N-1 \qquad (4.5)$$

$$u_N(\pm 1,t) = 0 \quad .$$

Note that (4.5) has boundary conditions in contrast to (4.4).

Let w_j be defined in (2.7). From (4.5) it is clear that

$$\sum_{j=0}^{N-1} \frac{u_N(x_j)}{1+x_j}\frac{\partial u_N}{\partial t}(x_j)\, w_j = \sum_{j=0}^{N-1}\frac{x_j}{1+x_j}u_N(x_j)\frac{\partial u_N}{\partial x}(x_j)\, w_j$$

$$= \sum_{j=0}^{N}\frac{x_j}{1+x_j}u_N(x_j)\frac{\partial u_N}{\partial x}(x_j)w_j - \left(\frac{\partial u_N}{\partial x}(-1)\right)^2$$

$$\le \int \frac{x}{(1+x)\sqrt{1-x^2}}\, u_N \frac{u_N}{x}\, dx = -\int \frac{u_N^2}{(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}(1-x+x^2) \le 0$$

and that proves stability.

It goes without saying that the proofs of the last two theorems can be extended to all the functions $c(x)$ such that $\frac{c(x)}{x}$ is of constant sign. We conjecture that it is true for any $c(x)$.

In Table I we show the results of applying the Chebyshev-pseudospectral method to four equations.

The first equation is

$$\begin{aligned} u_t &= (1+x)a_x \qquad |x| \leq 1 \qquad \text{(a)} \qquad (4.6) \\ u(x,0) &= \sin \pi x \\ u(1,t) &= \sin(2e^t-1)\pi \end{aligned}$$

The solution to this problem is

$$u(x,t) = \sin \pi[(1+x)e^t-1] \,. \qquad \text{(b)} \qquad (4.6)$$

This problem has a characteristic boundary at $x = -1$, moreover for large t the solution has a large variation in the neighborhood of $x = -1$.

The second problem is

$$\begin{aligned} u_t &= (1-x)u_x \qquad \text{(a)} \qquad (4.7) \\ u(x,0) &= \sin \pi x \,. \end{aligned}$$

The solution is given by

$$u(x,t) = \sin \pi[1-(1-x)e^{-t}] \,. \qquad \text{(b)} \qquad (4.7)$$

The line $x = 1$ is a characteristic boundary but in contrast to the equation (4.6) the neighboring characteristics point from the boundary towards the domain. The third is

$$u_t = x\, u_x \qquad \text{(a)} \qquad (4.8)$$

$$u(x,0) = \sin \pi x$$

$$u(1,t) = \sin(\pi e^t)$$

$$u(-1,t) = -\sin(\pi e^t)$$

The solution is

$$u(x,t) = \sin(\pi x e^t) \qquad \text{(b)} \qquad (4.8)$$

And the fourth is

$$u_t = -x\, u_x \qquad \text{(a)} \qquad (4.9)$$

$$u(x,0) = \sin \pi x$$

where

$$u(x,t) = \sin(\pi x e^{-t}) \qquad \text{(b)}$$

All these problems were solved by Chebyshev pseudospectral methods with the modified Euler time marching techniques. With the time step $\Delta t = 1/\Delta x_{min}$ we advance from the time 0 to the time $t = 2$. Note that since $x_N - x_{N-1} = 0\left(\frac{1}{N^2}\right)$ then $\Delta x_{min} \sim 0\left(\frac{1}{N^2}\right)$.

In Table I we show the L_2 Chebyshev errors of the solution of the problems (4.6) - (4.9). It is clear that the Chebyshev collocation method was stable for all these problems and has the same rate of convergence.

However, the errors for problems (4.6) and (4.8) were much larger than those of (4.7) and (4.9). In fact taking 64 modes in the solution of (4.6) and (4.8) produce the same error that 17 modes produce for (4.6) and (4.8). This is a problem of accuracy and not of stability. The question now is do we retain spectral accuracy? To answer this question we ran the problem (4.8) with smaller and smaller time steps until the results were not changed which means that we get the space accuracy. For 17 modes we got an L_2 error of $1.16.10^{-1}$, whereas for 33 modes we got an error of 6.10^{-5}. This indicates the fact that the order of accuracy in space is indeed better than any algebraic order.

Conclusion

It has been shown in this paper that the pseudospectral-Chebyshev methods are convergent in variable coefficient parabolic problems and in some cases to hyperbolic problems. The analysis shows that the rate of convergence is greater for finite difference methods or the finite element method. It seems that for a single first order hyperbolic equation the method remains stable even when the coefficient changes sign, though in this case care must be taken to have adequate spatial resolution. This fact, combined with the fact that collocation methods are easy to apply in the nonlinear case, shows that pseudospectral method is in general preferable to Galerkin or Tau methods.

REFERENCES

1. D. GOTTLIEB & S. A. ORSZAG, *Numerical Analysis of Spectral Methods, Theory and Applications*, NSF-CBMS Monogram, No. 26, SIAM (1977).

2. H. O. KREISS & J. OLIGER, *Methods for the Approximate Solution of Time Dependent Problems*, G.A.R.P. Publication Series, No. 10 (1973).

3. D. GOTTLIEB & E. TURKEL, "On time discretizations for spectral methods," ICASE Report 78-1, to appear in "*Studies in Applied Math.*"

4. D. GOTTLIEB, M. GUNZBURGER, and E. TURKEL, "On numerical boundary treatment for hyperbolic systems," ICASE Report 78-13, to appear in SIAM J. on Numer. Anal.

TABLE I. L_2 Chebyshev errors for the solution of (4.6) - (4.9).

N	$u_t = (1+x)u_x$	$u_t = (1-x)u_x$	$u_t = x u_x$	$u_t = -x u_x$
17	$1.13 \cdot 10^{-1}$	$9.4 \cdot 10^{-6}$	$1.16 \cdot 10^{-1}$	$2.05 * 10^{-6}$
33	$1.79 \cdot 10^{-3}$	$4.7 \cdot 10^{-7}$	$2.59 \cdot 10^{-4}$	$1.05 * 10^{-7}$
65	$8.5 \cdot 10^{-5}$	$2.2 \cdot 10^{-8}$	$1.22 \cdot 10^{-5}$	$5 * 10^{-9}$